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Automorphisms and Symmetries of Quantum Logics

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Given an amalgam of groups

$$G_0 \begin{array}{l} \nearrow G_1 \\ \searrow G_2 \end{array}$$

then every quantum logic $Q_0=(L_0, M_0)$ (L_0 is a σ-orthomodular poset, M_0 is a full set of states on it) satisfying some reasonable conditions can be embedded in a quantum logic $Q=(L, M)$, in which (1) all the automorphisms of L form a group $\simeq G_1$, (2) all the automorphisms of M form a group $\simeq G_2$, and (3) all the symmetries of Q form a group $\simeq G_0$. The quantum logic of all closed subspaces of a Hilbert space H and all its measures satisfies the conditions required from Q_0; hence, enlarging it, one can obtain "anything."

1. INTRODUCTION AND THE MAIN THEOREM

Every abstract group can be represented as the group of *all* automorphisms of an orthomodular lattice. This result of Kalmbach (1984) was enriched by the investigation of states in Kallus and Trnková (1987), where collections of quantum logics with some prescribed properties (representing prescribed groups by their symmetries and a prescribed order on the index set by the embeddability) were constructed.

Here, I investigate symmetries of quantum logics, automorphisms of the corresponding σ-orthomodular posets and the bijections of the set of states, preserving all the σ-convex combinations. I show that their connections are rather free in general; then can represent any amalgam of groups.

First, recall the appropriate names and notions. *A quantum logic* is a pair $Q=(L, M)$, where L is a *σ-orthomodular poset* [i.e., a partial order $\leq$ on L and a complementation $': L \to L$ are given such that L has the smallest element 0, the largest element 1, $0 \neq 1$, and $(p')'=p$, $p \vee p'=1$, $p \wedge p'=0$ for

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all $p \in L$, $p \leq q$ iff $p' \geq q'$, $p \leq q$ implies $q = p \vee (q \wedge p')$; moreover, if $p_1, p_2, \ldots$ is a sequence of pairwise orthogonal elements, i.e., $p_i \leq p_j'$ for $i \neq j$, then the join $\bigvee_{n=1}^{\infty} p_n$ exists in L] and M is a *σ-convex full set of states on* L [i.e., each $m \in M$ is a map of L into $\langle 0, 1\rangle$ such that $m(0) = 0$, $m(p') = 1 - m(p)$, and $m(\bigvee_{n=1}^{\infty} p_n) = \sum_{n=1}^{\infty} m(p_n)$ whenever $p_1, p_2, \ldots$ is a sequence of pairwise orthogonal elements; moreover, M is closed under the forming of σ-convex combinations, i.e., for any sequence $\{\alpha_n\}$ of real numbers and $\{m_n\}$ of states,

$$\alpha_n \geq 0 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \alpha_n = 1 \;\Rightarrow\; \sum_{n=1}^{\infty} \alpha_n m_n \in M$$

and M is full in the sense that it determines the order of L, i.e., for every $p, q \in L$,

$$\forall m \in M \qquad m(p) \leq m(q) \;\Rightarrow\; p \leq q]$$

A *sublogic* $Q_0(L_0, M_0)$ of a quantum logic $Q = (L, M)$ is determined by a one-to-one strong homomorphism $h: L_0 \to L$ [i.e., $x \leq y$ in L_0 iff $h(x) \leq h(y)$ in L, h preserves 0, complements, and the joins of pairwise orthogonal sequences] such that

$$\{m \circ h \mid m \in M\} = M_0$$

(i.e., L is an enlarging of L_0 and each state in M_0 is extended to L, not necessarily in a unique way; the set of these extensions is the state set M); clearly, the map

$$\bar{h}: \quad M \to M_0$$

given by $\bar{h}(m) = m \circ h$ which is required to be surjective by our definition, preserves the σ-convex combinations. We say that Q_0 *can be embedded in* Q if it is its sublogic in the above sense.

A symmetry of a quantum logic $Q = (L, M)$ (Pulmannová, 1977) is any automorphism $\tau: L \to L$ for which

$$\{m \circ \tau \mid m \in M\} = M$$

Clearly, all the symmetries of Q form a group; let us denote it by Aut Q. It is a subgroup of the group Aut L of all the automorphisms of L. The third group investigated here, associated with the quantum logic $Q = (L, M)$, is the group of all the bijections

$$b: \quad M \to M$$

which preserve the σ-convex combinations, i.e.,

$$b\left(\sum_{n=1}^{\infty} \alpha_n m_n\right) = \sum_{n=1}^{\infty} \alpha_n b(m_n)$$

$$\text{whenever} \quad m_n \in M, \quad \alpha_n \geq 0, \quad \sum_{n=1}^{\infty} \alpha_n = 1$$

Let us denote the last group by Aut M.

The preservation of the σ-convex combinations is equivalent to the preservation of the convex combinations, so that the elements of Aut M are the stochastic symmetries in the sense of Cook and Rüttimann (1985). These three groups form an amalgam in a natural way, namely

$$\begin{array}{ccc} & & \mathrm{Aut}\, L \\ & \overset{i}{\nearrow} & \\ \mathrm{Aut}\, Q & & \\ & \underset{\kappa}{\searrow} & \\ & & \mathrm{Aut}\, M \end{array}$$

where i is the inclusion map and the one-to-one homomorphism κ is given by the formula

$$[\kappa(\tau)](m) = m \circ \tau^{-1} \qquad \text{for all} \quad m \in M$$

Are there any other general relations among these groups? I prove here that the answer to this question is negative: any amalgam of groups

$$\mathscr{A}: \quad \begin{array}{ccc} & & G_1 \\ & \overset{h_1}{\nearrow} & \\ G_0 & & \\ & \underset{h_2}{\searrow} & \\ & & G_2 \end{array}$$

(i.e., G_0, G_1, and G_2 are arbitrary abstract groups, h_1 and h_2 are one-to-one homomorphisms) can be realized by a quantum logic $Q = (L, M)$, in the sense that there exist isomorphisms

Φ_0 of G_0 onto Aut Q
Φ_1 of G_1 onto Aut L
Φ_2 of G_2 onto Aut M

such that $i \circ \Phi_0 = \Phi_1 \circ h_1$ and $\kappa \circ \Phi_0 = \Phi_2 \circ h_2$. This is the first statement of our main theorem. Moreover, a quantum logic realizing a given amalgam $\mathscr{A}$ can be constructed such that it contains a given quantum logic $Q_0 = (L_0, M_0)$ (satisfying some conditions) as its sublogic—this shows that in fact a given amalgam can be realized in many distinct ways; or, on the other hand, that the quantum logic $Q_0 = (L_0, M_0)$ can be enlarged such that one obtains a previously prescribed amalgam.

Before the formulation of the Main Theorem, recall that a σ-orthomodular poset L_0 is called *atomist* if each $l \in L_0$ is a join of the set of all atoms $a \in L_0$ with $a \leq l$; and a state $m \in M_0$ is called *pure* if $m = \alpha m_1 + (1-\alpha) m_2$ with $0 < \alpha < 1$ implies $m_1 = m_2 = m$.

Main Theorem. Let $Q_0 = (M_0, L_0)$ be a quantum logic such that L_0 is atomistic and M_0 satisfies the following conditions:

(α) Every state in M_0 is a σ-convex combination of pure states.
(β) For every ordered pair a, b of distinct atoms of L_0 there exists a state $m \in M_0$ with $m(a) = 1 > m(b)$.
(γ) There exists a collection $\{s_a \mid a \in A\}$ of pure states of M_0, where A is the set of all atoms of L_0, such that (i) $(\forall a \in A)(s_a(a) = 0)$; (ii) $s_a(b) + s_b(s) < 2$ for every $a, b \in A$, $a \neq b$.

Then every amalgam of groups can be realized by a quantum logic $Q = (L, M)$, containing Q_0 as a sublogic. Moreover, L is also atomistic and if L_0 is a lattice or σ-complete lattice or a complete lattice, so is L.

Remark. The quantum logic $Q_0 = (L_0, M_0)$ of all closed subspaces of a separable complex Hilbert space H with dim ≥ 3 and all the σ-additive probabilities satisfies the requirements of the Main Theorem. It is well known that L_0 is atomistic and (α), (β) follow immediately from Gleason's theorem. However, the condition (γ) also follows from Gleason's theorem: it suffices only to find a map $\lambda: A \to A$ without 2-cycles such that a and $\lambda(a)$ are always orthogonal [since dim $H \geq 3$, for every $a \in A$ the set $O(a) \subseteq A$ of all atoms orthogonal to a is large enough: card $O(a) = 2^{\aleph_0}$; this makes it possible to construct such λ by the transfinite induction] and put $s_a = q_{\lambda(a)}$, where q_b is the pure state associated to the atom b. This is presented in Trnková (1988), where also the Main Theorem was announced. The proof of the Main Theorem appears for first time here. It is rather involved and it uses some graph-theoretic techniques. The most involved part of the proof is the construction of an embedding of the given quantum logic Q_0 into a rigid one.

2. QUANTUM LOGICS DETERMINED BY GRAPHS

1. Let us denote by $\mathfrak{G}$ the class of all undirected graphs (V, E) (i.e., V is the set of its *vertices*, not necessarily finite, E is the set of its *edges*, i.e., each $e \in E$ is a two-element subset of V) which are:

Without triangles (i.e., if $x, y, z \in V$, then at least one of the edges $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ is not in E).

Without squares (i.e., if $x, y, z, v \in V$, then at least one of the edges $\{x, y\}$, $\{y, z\}$, $\{z, v\}$, $\{v, x\}$ is not in E).

Of $\deg x \geq 2$ *for all* $x \in V$ (i.e., there exist distinct $y, z \in V$ with $\{x, y\}$, $\{x, z\} \in E$).

Of card $V \geq 5$ and for each $x \in V$ there exists $y \in V \setminus \{x\}$ with $\{x, y\} \notin E$.

As is well known (see. e.g., Kalmbach, 1983), every $G=(V, E)$ determines an orthomodular lattice $L(G)$ as follows: each edge $e=\{x, y\} \in E$ is cut into two edges and the obtained undirected graph is the Greechie diagram (Kalmbach, 1983) of $L(G)$. Informally, each edge $e=\{x, y\} \in E$ is replaced by a copy of these Boolean algebra 2^3 with three atoms $x, y, x' \wedge y'$; in all these Boolean algebras 0 and 1 are identified and, moreover, if two edges $e, \bar{e} \in E$ have a vertex x in common, say $e=\{x, y\}$, $\bar{e}=\{x, \bar{y}\}$, the corresponding Boolean algebras have the atom x and the coatom x' in common. Since $G=(V, E)$ has no triangles and no squares, $L(G)$ is really an orthomodular lattice (Kalmbach, 1983). All the Boolean blocks [=maximal Boolean algebras; see Kalmbach (1983)] of $L(G)$ are isomorphic to 2^3. In the convention that $x, y, x' \wedge y'$ are atoms of the Boolean block corresponding to $e=\{x, y\}$, we may suppose that

$$V \subseteq L(G).$$

Let us call each $x \in V$ *a vertex of* $L(G)$ and each x' *a covertex of* $L(G)$.

2. We say that a Boolean block B of a σ-orthomodular poset L is *clear* (Kallus and Trnková, 1987) if there exists an atom $a \in B$ such that there exists precisely two distinct elements of $L \backslash \{1\}$, say a_1, a_2 with $a < a_1$, $a < a_2$; then the atom a with this property is called *clear*, too.

If $G=(V, E) \in \mathfrak{G}$, then each Boolean block of $L(G)$ is clear, the clear atom of the block corresponding to $\{x, y\} \in E$ is $x' \wedge y'$, the covertices x', y' are the only elements of $L(G) \backslash \{1\}$ dominating it. Since $\deg x \geq 2$ for each $x \in V$, no x is a clear atom [in fact, if $\{x, y\}$ and $\{x, \bar{y}\}$ are distinct edges having the vertex x in common, then $x \vee y$, $x \vee \bar{y}$, y', $\bar{y}'$ are distinct elements of $L(G)$ dominating x].

Every automorphism γ of G determines uniquely the automorphism $L(\gamma)$ of $L(G)$ extending it, i.e.,

$$[L(\gamma)](x)=\gamma(x) \qquad \text{for all} \quad x \in V$$

because then, necessarily, $[L(\gamma)](x')=(\gamma(x))'$, $[L(\gamma)](x \vee y)=\gamma(x) \vee \gamma(y)$, and $[L(\gamma)](x' \wedge y')=(\gamma(x))' \wedge (\gamma(y))'$ for all $\{x, y\} \in E$ and 0 and 1 are fixed points of $L(\gamma)$.

Conversely, if τ is an automorphism of $L(G)$, then, necessarily, $\tau = L(\gamma)$ for some automorphism γ of G. In fact, τ sends the set of all clear atoms onto itself and the set of all the remaining atoms also onto itself. This implies that the domain-range restriction of τ maps bijectively V onto itself and, since τ maps each Boolean block onto a Boolean block, this restriction is an automorphism of G. We conclude that the groups Aut G and Aut $L(G)$ are isomorphic, the isomorphism is given by

$$\gamma \rightsquigarrow L(\gamma)$$

and the inverse isomorphism is the domain-range restriction only.

3. Let $G=(V, E)$ be in $\mathfrak{G}$, let m: $L(G)\to\langle 0, 1\rangle$ be a state, i.e.,

$$m(0)=0, \qquad m(1)=1$$

$$m(x')=1-m(x) \qquad \text{for each} \quad x\in V \tag{1}$$

$$m(x\vee y)=m(x)+m(y), \qquad m(x'\wedge y')=1-[m(x)+m(y)]$$

for each $\{x, y\}\in E$

Thus, the state m is determined by its values $m(x)$ with $x\in V$, the values at 0, 1 x', $x\vee y$, $x'\wedge y'$ are given by equations (1). Clearly,

$$\text{a map } m\text{:} \quad V\to\langle 0, 1\rangle \text{ determine a state on } L(G)$$
$$\text{iff } m(x)+m(y)\leq 1 \text{ for all } \{x, y\}\in E \tag{2}$$

Let I be an *independent set of the graph* $G=(V, E)\in\mathfrak{G}$ (i.e., $I\subseteq V$ and never $\{x, y\}\in E$ for $x, y\in I$). Then the state m_I is defined by

$$m_I(x)=1 \qquad \text{if } x\in I$$

$$m_I(x)=0 \qquad \text{if } x\in V\backslash I$$

(i.e., its restriction to V is the characteristic function χ_I of I in V). Since χ_I satisfies (2), m_I is really a state.

Observation. For each independent set I of G, m_I is a pure state on $L(G)$. (In fact, every two-valued state is a pure state.)

4. Let $\mathfrak{G}$ and $G=(V, E)\in\mathfrak{G}$, $L(G)$ be as in Sections 2.1–2.3. Let $\mathscr{I}$ be the set of *all* independent sets of G and, for each $I\in\mathscr{I}$, let m_I be the state as in Section 2.3. We say that $\mathscr{T}\subseteq\mathscr{I}$ is a *full system* of independent sets of G if

$$\emptyset\in\mathscr{T}, \quad \{x\}\in\mathscr{T} \quad \text{for all } x\in V, \text{ and}$$
$$\text{for every } x, y\in V \text{ with } \{x, y\}\notin E \text{ there exists } I\in\mathscr{T} \tag{3}$$
$$\text{such that } \{x, y\}\subseteq I$$

Lemma. Let $\mathscr{T}\subseteq\mathscr{I}$ be a full system of independent sets of G. Then $P_{\mathscr{I}}=\{m_I \mid I\in\mathscr{T}\}$ is a full set of states on $L(G)$.

Proof. If a, $b\in L(G)$, then

$$a\not\leq b \Rightarrow (\exists I\in\mathscr{T})(m_I(a)=1>0=m_I(b))$$

This can be proved by a staightforward verification discussing all the possible cases. We show, e.g., the example when a is a vertex and $b=x'\wedge y'$ for vertices x, y with $\{x, y\}\in E$: if $a\in\{x, Y\}$, then $m_{\{a\}}(a)=1>0=m_{\{a\}}(b)$; if $a\notin\{x, y\}$, then (since G does not contain triangles) either $\{a, x\}\notin E$ or

$\{a, y\} \not\subseteq E$; then there exists $I \in \mathcal{T}$ such that either $\{a, x\} \subseteq I$ or $\{a, y\} \subseteq I$, hence $m_I(a) = 1 > 0 = m_I(b)$. ■

5. Let $G = (V, E) \in \mathfrak{G}$, and let $\mathcal{J}$ be the set of all independent sets of G. We say that $\mathcal{T} \subseteq \mathcal{J}$ is *invariant* if for every $\gamma \in \operatorname{Aut} G$, the system $\gamma\mathcal{T} = \{\gamma(I) \mid I \in \mathcal{T}\}$ is equal to $\mathcal{T}$.

Let $\gamma \in \operatorname{Aut} G$; denote $\tau = L(\gamma) \in \operatorname{Aut} L(G)$. Clearly,

$$m_{\gamma(I)} = m_I \circ \tau^{-1}$$

Hence, if $\mathcal{T} \subseteq \mathcal{J}$ is invariant, then

$$\{m_I \mid I \in \mathcal{T}\} = \{m_I \circ \tau^{-1} \mid I \in \mathcal{T}\}$$

for every $\gamma \in \operatorname{Aut} G$, $\tau = L(\gamma) \in \operatorname{Aut} L(G)$.

6. Let $G = (V, E) \in \mathfrak{G}$, put $\mathcal{T}(G) = \{\emptyset\} \cup \{\{x\} \mid x \in V\} \cup \{\{x, y\} \mid x, y \in V, \{x, y\} \not\subseteq E\}$. Clearly, $\mathcal{T}(G)$ is an invariant full system of independent subsets of G. Put

$$P(G) = \{m_I \mid I \in \mathcal{T}(G)\}$$

and denote by $M(G)$ the σ-convex envelope of $P(G)$. Then

$$Q(G) = (L(G), M(G))$$

is a quantum logic; $P(G)$ is precisely the set of all pure states of $Q(G)$. Moreover, for every $\tau \in \operatorname{Aut} L(G)$,

$$\{m \circ \tau^{-1} \mid m \in M(G)\} = M(G)$$

so that each $\tau \in \operatorname{Aut}(L(G))$ is already a symmetry of the quantum logic $Q(G)$. Hence, we see that, in the amalgam,

$$\begin{array}{ccc} & & \operatorname{Aut}(L(G)) \\ & \overset{i}{\nearrow} & \\ \operatorname{Aut} Q(G) & & \\ & \underset{\kappa}{\searrow} & \\ & & \operatorname{Aut} M(G) \end{array}$$

The inclusion i is the identity. Now, we show that κ is also surjective.

7. *Proposition.* Let $G = (V, E)$ be in $\mathfrak{G}$, and let $b \in \operatorname{Aut} M(G)$. Then there exists $\tau \in \operatorname{Aut} L(G)$ such that

$$b(m) = m \circ \tau^{-1} \qquad \text{for all } m \in M(G)$$

Proof. I present here an elementary proof, which is instructive also for the proofs in the next parts.

(i) Since $P(G)$ is the set of all pure states of $M(G)$, b maps $P(G)$ onto itself. Hence, there is a bijection B of $\mathscr{T}(G)$ onto itself such that, for all $I \in \mathscr{T}(G)$,

$$b(m_I) = m_{BI}$$

To prove the Proposition, it suffices to show that

$$\begin{aligned} &B\phi = \varnothing \\ &B\{x\} \text{ is a one-point set, say } \{\bar{x}\} \\ &B\{x, y\} = \{\bar{x}, \bar{y}\} \end{aligned} \tag{4}$$

In fact, then B determines a bijection γ of V onto itself by the rule

$$\gamma(x) = \bar{x}$$

Moreover, if $\{x, y\} \notin E$, then $\{\gamma(x), \gamma(y)\} \notin E$. On the other hand, if $\{\tilde{x}, \tilde{y}\} \notin E$, then there is $I \in \mathscr{T}(G)$ with $BI = \{\tilde{x}, \tilde{y}\}$ because B is surjective. Then (4) implies that I has precisely two elements and $I \notin E$, so that

$$\{\tilde{x}, \tilde{y}\} \notin E \Rightarrow \{\gamma^{-1}(x), \gamma^{-1}(y)\} \notin E$$

Thus, γ is an automorphism of G. Clearly, $\tau = L(\gamma)$ satisfies $b(m_I) = m_{BI} = m_I \circ \tau^{-1}$ for all $I \in \mathscr{T}(G)$. Since b preserves the σ-convex combinations, we have

$$b(m) = m \circ \tau^{-1} \qquad \text{for all} \quad m \in M(G)$$

(ii) Thus, it suffices to prove (4). First, we show that $B\phi = \varnothing$. Let us suppose the contrary; let $z \in B\phi$. For any $x, y \in V$, $x \neq y$, $\{x, y\} \notin E$, we have

$$\tfrac{1}{2}m_{\{x\}} + \tfrac{1}{2}m_{\{y\}} = \tfrac{1}{2}m_{\{x,y\}} + \tfrac{1}{2}m_{\phi} \tag{5}$$

so that, since b preserves this convex combination,

$$\tfrac{1}{2}m_{B\{x\}} + \tfrac{1}{2}m_{B\{y\}} = \tfrac{1}{2}m_{B\{x,y\}} + \tfrac{1}{2}m_{B\phi} \tag{6}$$

Since $m_{B\phi}(z) = 1$, necessarily $z \in B\{x\}$ or $z \in B\{y\}$. Hence, if x, y are distinct vertices of G such that $z \notin B\{x\}$ and $z \notin B\{y\}$, necessarily $\{x, y\} \in E$. Since $G = (V, E)$ contains no triangle, necessarily $z \in B(x_i)$ for some x_i from each triple x_1, x_2, x_3 of distinct vertices of G. Since card $V \geq 5$, there are at least three distinct vertices y_1, y_2, y_3 such that $z \in B\{y_i\}$ for all $i = 1, 2, 3$. Since G contains no triangle, there exist such $x, y \in \{y_1, y_2, y_3\}$ that $\{x, y\} \notin E$. Hence, we have constructed a couple x, y of distinct vertices of G such that

$$z \in B\{y\}, \qquad z \in B\{y\} \qquad \text{and} \qquad \{x, y\} \notin E$$

Then equation (6) implies that $z \in B\{x, y\}$. Since B is a bijection, $B\{x\}$, $B\{y\}$, and $B\phi$ are four distinct elements of $\mathscr{T}(G)$ and equation (6) implies

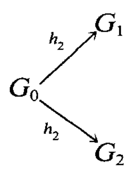
$B\{x\} \cup B\{y\} = B\{x, y\} \cup B\phi$. The elements of $\mathcal{T}(G)$ are subsets of V with at most two elements and the couple x, y was constructed such that the vertex z, which was supposed to be in $B\phi$, is also in $B\{x\}$, $B\{y\}$, $B\{x, y\}$. Hence $B\{x\}$, $B\{y\}$, $B\{x, y\}$, and $B\phi$ cannot be distinct, which is a contradiction. Consequently, $B\phi = \varnothing$.

(iii) Now, we finish the proof of (4). Equation (6) and $B\phi = \varnothing$ imply that for every pair x, y of distinct vertices x, y with $\{x, y\} \notin E$ we have $B\{x\} \cup B\{y\} = B\{x, y\}$. Since B is a one-to-one map, $B\{x\}$, $B\{y\}$, and $B\{x, y\}$ are three distinct elements of $\mathcal{T}(G)$, so that necessarily $B\{x\} = \{\bar{x}\}$, $B\{y\} = \bar{y}\}$, and $B\{x, y\} = \{\bar{x}, \bar{y}\}$ for some distinct $\bar{x}, \bar{y} \in V$ with $\{\bar{x}, \bar{y}\} \notin E$. Since, for every $x \in V$, there exists distinct $y \in V$ with $\{x, y\} \notin E$ (see the definition of $\mathfrak{G}$ in Section 2.1), the vertex $\bar{x}$ is determined for each $x \in V$. □

3. QUANTUM LOGICS REALIZING A GIVEN AMALGAM

1. Let an amalgam of groups

$\mathscr{A}$:



be given. Since we work with it up to isomorphism, we may suppose (see, e.g., Kurosh, 1957) that the amalgam is formed by subgroups of a group G, and h_1 and h_2 are inclusions, say

$$G_1 \subseteq G, \qquad G_2 \subseteq G, \qquad G_0 = G_1 \cap G_2$$

We are going to realize $\mathscr{A}$ by a quantum logic in the sense of Section 1. The quantum logic Q will be consructed based on the following data:

> three undirected graphs $H = (W, E)$, $H_1 = (W, E_1)$, $H_2 = (W, E_2)$ on the same set of vertices W, all of them in $\mathfrak{G}$, and a set $J \subseteq W$

such that all the following statements are satisfied:

(a) $E \subseteq E_1 \subsetneq E_2$.
(b) Aut $H_i \subsetneq$ Aut H for $i = 1, 2$.
(c) There is an isomorphism Ψ of G onto Aut H, which sends G_i onto Aut H_i for both $i = 1, 2$.
(d) J is an independent set of H_1, card $J \geq 5$ and
  (i) $\{x, y\} \in E_2 \backslash E_1 \Rightarrow \{x, y\} \subseteq J$
  (ii) $\gamma \in \text{Aut } H_1 \cap \text{Aut } H_2 \Rightarrow \gamma(J) = J$
(e) No H, H_1, H_2 contains a 7-cycle.

All the graphs H, H_1, H_2 have the same set of vertices W, hence Aut H, Aut H_1, Aut H_2 all are subgroups of the group of all permutations of W, so that (b) is meaningful. Since, by (c), Ψ sends G_i onto Aut H_i for both $i=1, 2$, it sends $G_0 = G_1 \cap G_2$ onto Aut $H_1 \cap$ Aut H_2.

The graphs H, H_1, H_2 and $J \subseteq W$ which satisfy all these requirements (a)-(e) are constructed in the Appendix. The property (e) is not used in this section, but it will be used in Section 4.

2. For each of the graphs H, H_1, H_2 we have the σ-orthomodular lattices $L(H)$, $L(H_1)$, $L(H_2)$. They have the same set of vertices and covertices Section 2.1); they differ only in the forming of $x \vee y$ and $x' \wedge y'$. Since $E \subseteq E_1 \subseteq E_2$, we have

$$L(H) \subseteq L(H_1) \subseteq L(H_2)$$

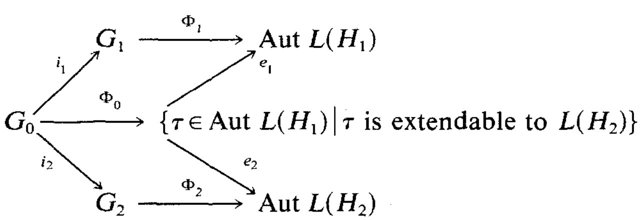
Every automorphism of each of them is of the form $L(\gamma)$, where γ is an automorphism of the corresponding graph; see Section 2.2. By (b) and (c), there exists an isomorphism Φ of G onto Aut $L(H)$ which sends G_i precisely on the group of the automorphisms of $L(H)$ extendable on $L(H_1)$, $i=1, 2$ (the such an extension is unique because H and H_i have the same set of vertices!). Notice that $\tau \in$ Aut $L(H)$ extendable to $L(H_2)$ can be unextendable to $L(H_1)$ because the restrictions of elements of Aut $L(H_2)$ need not be automorphisms of $L(H_1)$. On the other hand, the restriction of any $\tau \in$ Aut $L(H_1)$ to $L(H)$ is in Aut $L(H)$. This follows from assumption (b):

$$\text{Aut } H_1 \subseteq \text{Aut } H$$

Hence

$$\{\tau \in \text{Aut } L(H) \,|\, \tau \text{ is extendable both to } L(H_1) \text{ and to } L(H_2)\}$$
$$\cong \{\tau \in \text{Aut } L(H_1) \,|\, \tau \text{ is extendable to } L(H_2)\}$$

Hence the isomorphism Φ of G onto Aut $L(H)$ determines three isomorphisms Φ_0, Φ_1, and Φ_2 such that the following diagram commutes:



(7)

In (7), i_1, i_2, and e_1 are the inclusions, and $e_2(\tau)$ is the unique extension of τ. Moreover, if $\tau \in$ Aut $L(H_1)$ is extendable to $L(H_2)$, its restriction to

$L(H)$ is of the form $L(\gamma)$, where $\gamma \in \operatorname{Aut} H_1 \cap \operatorname{Aut} H_2$, so that

$$\tau(J) = J \tag{8}$$

by (d).

3. Since $L(H_1) \subseteq L(H_2)$ and $L(H_1)$ and $L(H_2)$ have the same set of vertices, the restriction of any state $m: L(J_2) \to \langle 0, 1\rangle$ to $L(H_1)$ is a state on $L(H_1)$ (se Section 2.3). Denote by P^+ the set of all the restrictions ${}^m/L(H_1)$, where $m \in P(H_2)$. Then we have isomorphisms

$$\operatorname{Aut} L(H_2) \xrightarrow{\kappa} \operatorname{Aut} M(H_2) \xrightarrow{r} \operatorname{Aut} M^+ \tag{9}$$

where M^+ is a σ-convex envelope of P^+, κ is the canonical isomorphism given by the formula $[\kappa(\tau)](m) = m \circ \tau^{-1}$, and $r(\tau)$ sends each ${}^m/L(H_1)$ to ${}^{\tau(m)}/L(H_1)$. Hence, $\operatorname{Aut} L(H_2)$, Φ_2, and e_2 can be replaced by $\operatorname{Aut} M^+$, $r \circ \kappa \circ \Phi_2$, and $r \circ \kappa \circ e_2$ in the diagram (7). On the other hand, M^+ is not a *full* set of states on $L(H_1)$ in general. [In fact, if $\{x, y\} \in E_2 \backslash E_1$, then x and y' are incomparable in $L(H_1)$, while $x \leq y'$ in $L(H_2)$, hence $m(x) > m(y')$ for *no* $m \in M^+$.] But

$$J = \{\phi\} \cup \{\{x\} \mid x \in W\} \cup \{\{x, y\} \mid x, y \in W, x \neq y, \{x, y\} \notin E_2\} \cup \{J\}$$

is a full system of independent sets of H_1 (see Section 2.3), by (d), so that $P^+ \cup [m_J = m_I \mid I \in J]$ is a full set of states on $L(H_1)$ (see Section 2.4). Let us denote by M the σ-convex envelope of $P^+ \cup \{m_J\}$, i.e., of $M^+ \cup \{m_J\}$. Then

$$Q = (L(H_1), M)$$

is a quantum logic which realizes the given amalgam. This follows immediately from (7), (9), and Lemmas A and B below [Lemma A implies $\operatorname{Aut} M^+ \simeq \operatorname{Aut} M$, Lemma B implies that $\operatorname{Aut} Q \simeq \{\tau \in \operatorname{Aut} L(H_1) \mid \tau$ is extendable to $L(H_2)\}$].

4. *Lemma A*. For every $b \in \operatorname{Aut} M$, $b(m_J) = m_J$ and b sends P^+ onto itself.

Proof. Since $P = P^+ \cup \{m_J\}$ is the set of all pure states of M, b sends it onto itself. For each $m \in P^+$ there exist states m_1, m_2, m_3 in P such that all the states m, m_1, m_2, m_3 are distinct and

$$\tfrac{1}{2}m + \tfrac{1}{2}m_1 = \tfrac{1}{2}m_2 + \tfrac{1}{2}m_3$$

[see equation (5) in Section 2.7]. Since card $J \geq 5$, m_J is the unique state in P which fails to satisfy it. Consequently $b(m_J) = m_J$. ■

Lemma B. For every $\tau \in \operatorname{Aut} L(H_1)$ extendable to $L(H_2)$,

$$m_J \circ \tau^{-1} = m_J$$

Proof. This follows from (8). ■

4. EMBEDDINGS INTO RIGID QUANTUM LOGICS AND THE PROOF OF THE MAIN THEOREM

1. Let us say that a quantum logic $Q=(L, M)$ is rigid, Aut $L\simeq\{1\}$ and Aut $M\simeq\{1\}$ (hence Aut $Q\simeq\{1\}$), where $\{1\}$ denotes the trivial group. Let us suppose that a given quantum logic $Q_0=(L_0, M_0)$ satisfies the assumption of the Main Theorem, i.e., L is atomistic and (α), (β), and (γ) are satisfied. In this section we show that Q_0 can be embedded in a rigid quantum logic. We can suppose that L_0 contains no blocks isomorphic to 2^2 and no clear block (see Section 2.1) isomorphic to 2^3. This follows easily from the following lemma.

Lemma. Let $Q_0(L_0, M_0)$ satisfy the requirements of the Main Theorem. Then it can be embedded into a quantum logic $Q_1=(L_1, M_1)$ such that Q_1 also satisfies the requirements of the Main Theorem and L_1 does not contain 2^2-blocks and clear 2^3-blocks.

Proof. The statement (γ) avoids 2^2-blocks. If B is a clear 2^3-block with atoms x, y, and z, and $z<v$ only for $v=x'$, y', or 1, we add two new atoms, say w and t, and split z as $z=w\vee t$ (hence, we add also coatoms w' and t' and $z'=w'\wedge t'$, $x\vee t=y'\wedge w', \ldots$), so that B is enlarged to a 2^4-block. Every state $m\in M_0$ is extended in two states m_w and m_t by

$$m_w(w)=m(z)=m_t(t), \qquad m_w(t)=0=m_t(w)$$

Let us denote by $\tilde{M}_0$ the σ-convex envelope of $\{m_w, m_t \mid m\in M_0\}$. Then the restriction on L_0 of all states in M_0 is precisely M_0 and a state $p\in\tilde{M}_0$ is a pure state in $\tilde{M}_0$ iff $p=m_w$ or $p=m_t$ for a pure state m in M_0. Clearly, (α) and (β) from the Main Theorem are fulfilled for $\tilde{M}_0$. We show that also (γ) is fulfilled. The new set of atoms A is equal to $(A\backslash\{z\})\cup\{w, t\}$. For $a\in A\backslash\{z\}$, put $t_a=(s_a)_w$ and put $\tilde{s}_w=(s_z)_w=s_z)_t=\tilde{s}_t$ [since $s_z(z)=0$, $s_z(w)=s_z(t)=0$]. Then $\{\tilde{s}_a \mid a\in A\}$ satisfies (γ).

Repeating this procedure for all clear 2^3-blocks in L_0, we obtain (L_1, M_1). ■

2. By the lemma, we suppose that the given quantum logic $Q_0=(L_0, M_0)$ satisfies the requirements of the Main Theorem and L_0 contains no 2^2-blocks and no clear 2^3-blocks. Let us recall that A denotes the set of all atoms of L_0, P the set of all pure states of M_0, and $\{s_1 \mid a\in A\}$ is as in (γ). We are going to construct a rigid quantum logic $Q=(L, M)$ containing Q_0. We construct L analogously as in Kallus and Trnková (1987): we find a suitable sufficiently large graph $G=(V, E)$ in $\mathfrak{G}$ (see Section 2.1) (for the

present construction, G has to be chosen in a rather special way; this will be described below) and a one-to-one map c of A into V such that $c(A)=\{c(a) \mid a \in A\}$ is an independent set of G. Then L is a σ-orthomodular poset obtained from $L_0 \cup L(G)$ by identifying 0 in L_0 with 0 in $L(G)$, 1 in L_0 with 1 in $L(G)$, and putting

$$a < (c(a))' \qquad \text{for all} \quad a \in A$$

[I.e., we add the elements $a \vee c(a)$ and $a' \wedge (c(a))'$ as in Kallus and Trnková (1987). Informally, in the horizontal sum $L_0 \dot{\cup} L(G)$, we join each a with $c(a)$ by a 2^3-block.] Clearly, if L_0 is a lattice or a σ-complete or complete lattice, so is L.

Let us suppose that Aut $G \simeq$ Aut $L(G)$ is trivial. We prove that also Aut L is trivial, as in Kallus and Trnková (1987). In fact, L_0 contains no clear 2^3-blocks, while every vertex of G is an atom of L, which is contained *only* in clear 2^3-blocks and there are at least two distinct clear blocks containing it. This property characterizes all the elements of L which are vertices of $L(G)$. Since any automorphism $\tau: L \to L$ has to preserve this property, it maps the set of all vertices of $L(G)$ onto itself, hence it maps $L(G)$ onto itself and its domain-range restriction to $L(G)$ is of the form $L(\gamma)$, where $\gamma \in$ Aut G (see Section 2.2). Since Aut $G \simeq \{1\}$, τ is the identity on $L(G)$. In particular, $\tau(c(a))=c(a)$, so that $\tau(a)=a$ for all $a \in A$. Since L_0 is atomistic, τ is the identity map, i.e., Aut $L \simeq \{1\}$.

3. To define the set M of states of a quantum logic Q with the required properties is more delicate. For every pure state $p \in P$ on L_0 we find a connected graph $G_p=(V_p, E_p)$ in $\mathfrak{G}$ such that:

(a) card $V_p > 5 + \text{card } P \times A$.
(b) G_p contains an independent set I_p with card $I_p >$ card A.
(c) $\{G_p \uparrow p \in P\}$ is a stiff collection of graphs (in the sense that if p_1, $p_2 \in P$ and there is an isomorphism γ of G_{p_1} into G_{p_2}, then necessarily $p_1 = p_2$ and γ is the identity).
(d) Every vertex of each G_p lies on a 7-cycle (this assumption is not used for the embedding of a given quantum logic into a rigid one, but for putting together the constructions of Section 3 and 4).

Concerning the existence of such a collection of graphs, see the Appendix.

We may suppose that $V_{p_1} \cap V_{p_2} = \varnothing$ whenever $p_1 \neq p_2$. We put

$$V = \bigcup_{p \in P} V_p, \qquad E = \bigcup_{p \in P} E_p$$

and use the graph $G=(V, E)$ in the construction described in Section 2. We have also to specify how $c(a)$ are chosen in V. Let us recall that

$\{s_a \mid a \in A\}$ is the collection of pure states on L_0, which satisfies the requirement (γ) of the Main Theorem. We choose

$$x_a = c(a) \text{ in } V_{s_a} \text{ such that if } a, b \in A, a \neq b, \text{ then } \{x_a, x_b\} \notin E$$

[If $s_a \neq s_b$, then this is satisfied automatically because no vertex of V_{s_a} is joined with any vertex of V_{s_b}; if $s_a = s_b$, then it is possible by (b).]

Since $\{G_p \mid p \in P\}$ is a stiff collection of connected graphs, G is a rigid graph, so that Aut $G \simeq$ Aut $L = \{1\}$, as shown in the previous section.

4. I describe how pure states on L_0 are extended on L. For each $p \in P$ put

$$R_p = \{\varnothing\} \cup \{\{x\} \mid x \in V_p\} \cup \{\{x, y\} \mid x, y \in V_p, x \neq y, \{x, y\} \notin E_p\}$$

and denote by

$$M_p = \{p_r \mid r \in R_p\}$$

the set of extensions of p defined as follows:

$$\begin{aligned} &p_r \text{ restricted to } L_0 \text{ is } p \\ &p_r \text{ restricted to } V_p \text{ is the characteristic function of } r \\ &\quad [\text{i.e., } p_r(z) = 1 \text{ if } z \in r;\ p_r(z) = 0 \text{ for all } z \in V_p \setminus r] \\ &p_r(z) = \tfrac{1}{4} \text{ for all } z \in V \setminus (V_p \cup \{x_a \mid a \in A\}) \\ &p_r(x_a) = 1 - \max(\tfrac{3}{4}, \tfrac{1}{2}[1 + p(a)]) \text{ whenever } a \in A \text{ and } p \neq s_a \end{aligned} \tag{10}$$

Lemma. For every $m \in M_p$, we have

$$\begin{aligned} m(l) &= p(l) && \text{for all} \quad l \in L_0 \\ m(x) + m(y) &\leq 1 && \text{whenever} \quad \{x, y\} \in E \\ m(x'_a) &\geq p(a) && \text{for all} \quad a \in A \end{aligned}$$

so that each $m \in M_p$ is really a state on L.

Proof. The first two statements are evident; let us show the last one. Let $m = p_r$ with $r \in R_p$. If $p \in P \setminus \{s_a\}$, then $p_r(x'_a) = \max(\frac{3}{4}, \frac{1}{2}[1 + p(a)]) \geq p(a) = p_r(a)$. If $p = a_a$, then $p_r(x'_a) \geq s_a(a)$ because $s_a(a) = 0$ by (γ) in the Main Theorem. ■

Lemma. If $p, q \in P$, $p \neq q$, then

$$\begin{aligned} p_r(z) &\leq \tfrac{1}{4} && \text{for all} \quad z \in V_q \text{ and all } r \in R_p \\ p_r(z) &= \tfrac{1}{4} && \text{for at least 5 elements } z \text{ of } V_q \end{aligned} \tag{11}$$

Proof. The first statement follows immediately from (10); the second follows from (10) and the fact that card $V_p > 5 + \text{card } A$ [see (a) in Section 3.3], so that $\text{card}(V_p \backslash \{x_a \mid a \in A\}) > 5$. ■

5. Put $\tilde{M} = \bigcup_{p \in P} M_p$.

Lemma. $\tilde{M}$ is a *full* set of states on L.

Proof. One has to show that if $l_1, l_2 \in L$ and $l_1 \not\leq l_2$, then there exists $m \in M$ such that $m(l_1) > m(l_2)$. The verification of this fact is quite easy in all the possible cases. However, *many* cases have to be discussed. I omit this long and tedious discussion and show the statement only in "the worst case" when $l_1 = a' \wedge x'_a$ and $l_2 = b \vee x_b$ for some $a, b \in A$, $a \neq b$. We need (γ) of the Main Theorem: either $s_a(b) < 1$ or $s_b(a) < 1$. If $s_a(b) < 1$, we choose $m = (s_a)_r$ with $r = \phi$. The $m(l_1) = 1 - [m(a) + m(x_a)] = 1 - [s_a(a) + 0] = 1$, while

$$m(l_2) = m(b) + m(x_b) = s_a(b) + 1 - \max(\tfrac{3}{4}, \tfrac{1}{2}[1 + s_a(b)]) < 1$$

If $s_b(a) < 1$, we choose $m = (s_b)_r$ with $r = \phi$. Then

$$m(l_1) = 1 - \{s_b(a) + 1 - \max(\tfrac{3}{4}, \tfrac{1}{2}[1 + s_b(a)]\}$$
$$= \max(\tfrac{3}{4}, \tfrac{1}{2}[1 + s_b(a)]) - s_b(a) > 0$$

while $m(l_2) = s_b(b) + 0 = 0$. ■

6. The set $\tilde{M} = \bigcup_{p \in P} M_p$ is a full set of states on L. Let M be its σ-convex envelope. Since M_0 is a σ-convex envelope of P, by (α) of the Main Theorem, the restriction on L_0 of any $m \in M$ belongs to M_0. Thus,

$Q = (L, M)$ is a quantum logic containing $Q_0(L_0, M_0)$ as a sublogic

Lemma. $\tilde{M}$ is precisely the set of all pure states of Q.

Proof. Let some $m = p_r$, $p \in P$, $r \in R_p$ be expressed as $m = \alpha m_1 + (1 - \alpha) m_2$ for some $0 < \alpha < 1$ and $m_1, m_2 \in M$. Since p_r is an extension of a pure state p, necessarily the restrictions of m_1 and of m_2 on L_0 are equal to p, so that necessarily both m_1 and m_2 are σ-convex combinations of states from M_p. Since $p_r(z) = 1$ for all $z \in r$ and $p_r(z) = 0$ for all $z \in V_p \backslash r$, necessarily $m_1 = m_2 = p_r$. Hence, every element of M is a pure state of M. Since M is the set of all σ-convex combinations of elements of M, no element of $M \backslash \tilde{M}$ is a pure state of M. ■

7. To prove that $Q = (L, M)$ is rigid, we have to show Aut $M \simeq \{1\}$. Thus, let b be in Aut M; we are going to prove that b is the identity. Since $\tilde{M}$ is the set of all pure state of M, b maps $\tilde{M}$ onto itself. For every $p \in P$ and $r \in R_p$, let us find $p^r \in P$ and $\tilde{r} \in R_{p^r}$ such that

$$b(p_r) = (p^r)_{\tilde{r}}$$

in the notation of (10). Let us write p_0, p_x, $p_{x,y}$ for p_r, and $p_{\bar{0}}^0$, $p_{\tilde{x}}^x$, $p_{x,y}^{x,y}$ for $p_{\tilde{r}}^r$ if $r = \emptyset$, $\{x\}$, $\{x, y\}$. For every $p \in P$ and every $x, y \in V_p$, $x \neq y$, $\{x, y\} \notin E_p$, we have

$$\tfrac{1}{2}p_x + \tfrac{1}{2}p_y = \tfrac{1}{2}p_{x,y} + \tfrac{1}{2}p_0$$

This equation has to be preserved by b, so we obtain

$$\tfrac{1}{2}p_{\tilde{x}}^x + \tfrac{1}{2}p_{\tilde{y}}^y = \tfrac{1}{2}p_{x,y}^{x,y'} + \tfrac{1}{2}p_{\bar{0}}^0 \tag{12}$$

8. *Proposition.* For every $p \in P$, $\tilde{\phi} = \emptyset$.

Proof. Let us suppose the contrary, i.e., there exists $p \in P$ such that $\tilde{\phi} \neq \emptyset$; choose $z \in \tilde{\phi} \subseteq V_{p_0}$. This p and z will be fixed during the whole proof, which is divided into several lemmas.

Lemma. $p^x = p^0 = p^y = p^{x,y}$ for all $x, y \in V_p$, $x \neq y$, $\{x, y\} \notin E_p$.

Proof. Denote

$$m_1 = \tfrac{1}{2}p_{\tilde{x}}^x + \tfrac{1}{2}p_{\tilde{y}}^y, \qquad m_2 = \tfrac{1}{2}p_{x,y}^{x,y} + \tfrac{1}{2}p_{\bar{0}}^0$$

By (12), $m_1 = m_2$, Since $z \in \tilde{\phi}$, we have $p_{\bar{0}}^0(z) = 1$, so that $m_2(z) \geq \frac{1}{2}$. If $p^x \neq p^0 \neq p^y$, then $m_1(z) \leq \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$, by (11). Hence either $p^x = p^0$ or $p^y = p^0$. Let us suppose that $p^x = p^0$. We prove that also $p^y = p^0$. If $p^y \neq p^x = p^0$, then $m_1(v) \geq \frac{1}{2} \cdot \frac{1}{4}$ for at least five elements v of $V_{\tilde{p}}^0$, by (11) again. Since the same has to be true also for m_2 and $\operatorname{card}(\{x, y\} \cup \tilde{\phi}) \leq 4$, necessarily $p^{x,y}$ is also different from $p^0 = p^x$. Then, for $v \in V_p^0$, $m_1(v) \geq \frac{1}{2}$ iff $v \in \{\tilde{x}\}$ and $m_2(v) \geq \frac{1}{2}$ iff $v \in \tilde{\phi}$. Consequently, $\{\tilde{x}\} = \tilde{\phi}$, so that $p_{\tilde{x}}^x = p_{\bar{0}}^0$. This is a contradiction, because $p_x \neq p_0$ and b is one-to-one. Thus, $p^y = p^x = p^0$. If $p^{x,y} \neq p^0$, then $m_2(v) \geq \frac{1}{2} \cdot \frac{1}{4}$ for at least five elements v of V_{p_0}, but $m_1(v) = 0$ except, possibly, four elements of V_{p_0}. This is a contradiction, hence $p^{x,y}$ is also equal to p^0. ■

Lemma. There exist $x, y \in V_p$, $x \neq y$, $\{x, y\} \notin E_p$ such that

$$z \in \{\tilde{x}\}, \qquad z \in \{\tilde{y}\}, \qquad z \in \{x, y\}$$

Proof. Since $p^x = p^y = p^{x,y} = p^0$ and $G_p = (V_p, E_p)$ is in $\mathfrak{G}$, we can proceed as in the proof of Section 2.7. ■

Now, we can finish easily the proof of the Proposition, as Section 2.7. Since $p^0 = p^x = p^y = p^{x,y}$ and z is in all $\tilde{\phi}$, $\{\tilde{x}\}$, $\{\tilde{y}\}$, and $\{x, y\}$, the elements in (12) cannot be all different, which is a contradiction.

9. As a consequence of Proposition 8, we obtain that, for all $p \in P$, x, $y \in V_p$, $x \neq y$, $\{x, y\} \notin E_p$, we have

$$\tilde{\phi} = \emptyset, \qquad \{\tilde{x}\} \neq \emptyset, \qquad \{\tilde{y}\} \neq \emptyset, \qquad \{x, y\} \neq \emptyset$$

In fact, $\tilde{\phi} = \varnothing$ is just the statement of Proposition 8. The statements $\{\tilde{x}\} \neq \varnothing$, $\{\tilde{y}\} \neq \varnothing$, $\{x, y\} \neq \varnothing$ follow also from Proposition 8, applied to $b^{-1} \in \operatorname{Aut} M$.

Let $p \in P$ be given. Choose $\bar{x} \in \{\tilde{x}\}$. Since $p_{\tilde{x}}^{x}(\bar{x}) = 1$, necessarily [by (11) and (12)], $p^{x} = p^{x,y}$ and $\bar{x} \in \{x, y\}$. Analogously, choosing $\bar{y} \in \{\tilde{y}\}$, we obtain that $p^{y} = p^{x,y}$ and $\bar{y} \in \{x, y\}$. Since $p^{x} = p^{x,y} = p^{y}$ but $p_{\tilde{x}}^{x} \neq p_{\tilde{y}}^{y}$, necessarily $\{\tilde{x}\} \neq \{\tilde{y}\}$; hence, we can choose $\bar{x} \neq \bar{y}$; since $\operatorname{card}\{x, y\} \leq 2$, necessarily $\{x, y\} = \{\bar{x}, \bar{y}\}$. Then (11) and (12) imply that $p^{0} = p^{x,y}$ [otherwise $p_{\tilde{0}}^{p}(v) = \frac{1}{4}$ for at least five elements of $V_{p^{x,y}}$, while $\frac{1}{2}p_{\tilde{x}}^{x} + \frac{1}{2}p_{\tilde{y}}^{y}$ would have the values on $V_{p^{x,y}}$ equal to 0 except possibly four elements]. Hence $p^{0} = p^{x} = p^{y} = p^{x,y}$ and, since $\tilde{\phi} = \phi$, we have

$$\{\tilde{x}\} = \{\bar{x}\}, \qquad \{\tilde{y}\}, \qquad \{x, y\} = \{\bar{x}, \bar{y}\}$$

We conclude that every $b \in \operatorname{Aut} M$ determines uniquely (1) a map of P into P given by $p \rightsquigarrow p^{0}$ ($= p^{x}$ for all $x \in V_p$), and (2) a map of each V_p into V_{p_0} given by $x \rightsquigarrow \bar{x}$, such that, if $x, y \in V_p$, $x \neq y$, and $\{x, y\} \notin E_p$, then $\{\bar{x}, \bar{y}\} \notin E_{p_0}$ and

$$b(p_0) = p_0^0, \qquad b(p_x) = p_{\bar{x}}^0, \qquad b(p_y) = p_{\bar{y}}^0, \qquad b(p_{x,y}) = p_{\bar{x},\bar{y}}^0$$

Since $b^{-1} \in \operatorname{Aut} M$ determines just the inverse maps, they all have to be one-to-one and

$$x \rightsquigarrow \bar{x}$$

is an isomorphism of $G_p(V_p, E_p)$ onto $G_{p^0} = (V_{p^0}, E_{p^0})$. Since $\{G_p \mid p \in P\}$ is a stiff collection of graphs, necessarily

$$p = p^0 \quad \text{for all} \quad p \in P \quad \text{and} \quad x \rightsquigarrow \bar{x} \quad \text{is the identity of } G_p$$

Thus, v: $M \to M$ is the identity. ■

10. The proof of the Main Theorem is already quite simple. Let H, H_1, H_2 be graphs as Section 3.1; let $\{G_p \mid p \in P\}$ be the collection of graphs as in Section 4. Choose $q \in P$ and define

$$G_q^1 = (V_q \cup W, E_q \cup E_1)$$

$$G_q^2 = (V_q \cup W, E_q \cup E_2)$$

(where we suppose $V_p \cap W = \phi$ for all $p \in P$). Choose all x_a with $s_a = q$ in V_q (never in W!). Define L by means of $\{G_p \mid p \in P \backslash \{q\}\} \cup \{G_q^1\}$ as in Section 4.2. Since every vertex of each G_p lies on a 7-cycle while H_1 does not contain any 7-cycle, and $\tau \in \operatorname{Aut} L$ maps $V = \bigcup_{p \in P} V_p$ itself. Hence $\operatorname{Aut} L \simeq \operatorname{Aut} H_1$. To define the set M of states, use the collection $\{G_p \mid p \in P \backslash \{q\}\} \cup \{G_q^2\}$ and proceed as in Sections 4.2–4.5; restrict only all these states to L (as in Section 4.3) and add the state q_J defined such that

q_J restricted to L_0 is q

q_J restricted to V_q W is the characteristic function of J

q_J is defined on V_p, $p \neq q$, as in (10)

and form the σ-convex envelope.

In the investigation of the state automorphisms $b \in \text{Aut } M$, first prove that

$$b(q_J) = q_J$$

(In fact, q_J is the unique pure state of M, for which there do not exist pure states m_1, m_2, m_3 such that all four states q_J, m_1, m_2, m_3 are distinct and

$$\tfrac{1}{2}q_J + \tfrac{1}{2}m_1 = \tfrac{1}{2}m_2 + \tfrac{1}{2}m_3$$

because card $J \geq 5$.) Then, proceeding as in Sections 4.7–4.9, we prove that every $b \in \text{Aut } M$ defines maps

$$P \text{ onto } P, \quad \text{by } p \rightsquigarrow p^0$$

$$W_p \text{ onto } W_{p^0}, \quad \text{by } x \rightsquigarrow \bar{x}$$

where $W_p = V_p$ if $p \neq q$, $W_q = V_q \cup W$. Since $\{G_p \mid p \in P\}$ is a stiff collection, every vertex of each G_p lies on a 7-cycle, while H_2 contains no 7-cycle; we conclude that

$p = p^0$ for all $p \in P$

$x = \bar{x}$ for all $x \in V_p$

$x \rightsquigarrow \bar{x}$, $x \in W$, has to be an isomorphism of H_2 onto itself

This gives $\text{Aut } M \simeq \text{Aut } H_2$, $\text{Aut } Q \simeq \text{Aut } H_1 \cap \text{Aut } H_2$, as in Section 3.

Remark. As can be seen from the construction, $Q = (L, M)$ is not strongly full [in the sense that $(m(l_1) = 1 \Rightarrow m(l_2) = 1) \Rightarrow l_1 \leq l_2$] whenever $Q_0 = (L_0, M_0)$ is strongly full. I do not know whether this strengthening of the Main Theorem is valid.

APPENDIX: GRAPH CONSTRUCTIONS

I. Let subgroups G_1, G_2 of a group G be given. I show how graphs $H = (W, E)$, $H_1 = (W, E_1)$, $H_2 = (W, E_2)$, and $J \subseteq W$, which satisfy (a)–(e) in Section 3.1 are constructed.

By Trnková (1986), there exists a directed graph (X, R) (i.e., $R \subseteq X \times X$) and R_1, $R_2 \subseteq R$, such that card $R \geq 5$ and:

(α) (X, R) is a connected graph without loops [i.e., never $(x, x) \in R$], and there is an isomorphism φ of $\text{Aut}(X, R)$ onto G.

(β) All the $\tau \in \text{Aut}(X, R)$ such that $(x, y) \in R_i$ iff $(\tau(x), \tau(y)) \in R_i$ form a group, say $\text{Aut}(X, R, R_i)$, which is sent by φ onto G_i.

We use the undirected graph K shown in Figure 1. The graph K consists of two complete 8-cycles plus one 8-cycle lacking the edge $\{a, b\}$; and two complete 6-cycles plus one 6-cycle lacking the edge $\{d, c\}$; the named vertices x, y will play a special role. We obtain $H = (W, E)$ from (X, R) such that each arrow $r = (x, y) \in R$ is replaced by a copy K_r of the graph K. In further detail, in the disjoint union $\bigcup_{r \in R} K_r$ we identify

x in K_r with x in $K_{\tilde{r}}$ whenever $\pi_1(r) = \pi_1(\tilde{r})$

x in K_r with y in $K_{\tilde{r}}$ whenever $\pi_1(r) = \pi_2(\tilde{r})$

y in K_r with y in $K_{\tilde{r}}$ whenever $\pi_2(r) = \pi_2(r) = \pi_2(\tilde{r})$

where $\pi_1(a, b) = a$, $\pi_2(a, b) = b$. We may suppose that $X \subseteq W$; X is just the set of all "gluing points' in the above 'arrow construction." In this sense, every $\tau \in \text{Aut}(X, R)$ can be extended (uniquely) to $\bar{\tau} \in \text{Aut}\, H$: $\bar{\tau}$ sends the whole copy K_r "identically" onto the copy $K_{\tilde{r}}$, where $r = (x, y)$, $\tilde{r} = (\tau(x), \tau(y))$. Conversely, every $\xi \in \text{Aut}\, H$ is the extension of some $\tau \in \text{Aut}(X, R)$. [In fact, ξ has to send z of each copy K_r on the point z of $K_{\tilde{r}}$ because z is the unique point which lies in an 8-cycle and has degree equal to 4; then u is the unique point which lies on a 8-cycle and with degree equal to 3, this implies that ξ has to send the points u on the points u, hence the whole 8-cycle of K_r "identically" on its copy in some $K_{\tilde{r}}$; similar

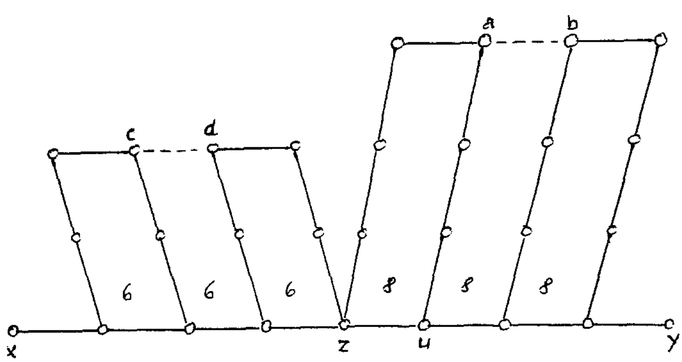


Fig. 1

reasoning show that ξ must map the whole K_r "identically" onto the copy $k_{\bar{r}}$—hence the restriction of $\xi \in \text{Aut}\, H$ to X is an element of Aut(X, R).] Consequently, Aut $H \simeq$ Aut(X, R).

The graph $H_1(W, E_1)$ is obtained from $H = (W, E)$ such that we add the edge $\{a, b\}$ to all the copies K_r with $r \in R_1$. Then $E \subseteq E_1$ and Aut $H_1 \simeq$ Aut$(X, R, R_1) \simeq G_1$. The graph $H_2 = (W, E_2)$ is obtained from $H = (W, E)$ such that we add the edge $\{a, b\}$ to all the copies K_r, $r \in R$, and we add also the edge $\{c, d\}$ to all those copies K_r with $r \in R_2$. Then $E_1 \subseteq E_2$ and Aut $H_2 \simeq$ Aut$(X, R, R_2) \simeq G_2$. Finally, J consists of all c and d of the copies K_r with $r \in R_2$ and of those vertices a and b which are in the copies K_r with $r \in R \backslash R_1$; moreover, for J to be large enough, we add the vertices z of all the copies K_r. The graphs H, H_1, H_2 contain no 7-cycle.

II. The collection $\{G_p \mid p \in P\}$ with the properties (a)–(d) of Section 4.3 can be taken from Pultr and Trnková (1980), where much stronger results are presented.

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